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Isomorphism of restricted chain-like graphs

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Abstract

We consider the isomorphism problem for the following set of graphs L :
for any graph $H \in L$, H can be decomposed by partitions of nodes $V_0, V_1, \cdots, V_m$ such that
(1) $|V_i| \leq k$ for each $0 \leq i \leq m$, $V(H) = \bigcup_{0 \leq i \leq m} V_i$, $V_i \cap V_j = \emptyset$, $i \neq j$,
(2) there no exist edges $\{x, y\}$ for any $x \in V_i$, $y \in V_j$, and $0 \leq i < j \leq m$, $j - i \geq 2$,
(3) the subgraph induced by V_i is connected for each $0 \leq i \leq m$.
In this paper, we will show that the isomorphism problem for L can be solved in $O(n^3)$ time.

1 Introduction

The problem of finding an efficient algorithm for testing whether two graphs are isomorphic is of fundamental importance the graph theory. Many classes of sets of graphs are investigated such as planar graphs [5], interval graphs [7], bounded degree graphs [8], and partial k-trees [1]. Bodlaender shows that for partial k-trees the isomorphism problem can be solved in $O(n^{k+4.5})$ time [1]. The result leads the fact that for k bounded bandwidth graphs the isomorphism problem can be solved in $O(n^{k+4.5})$ time.

We focus our attention to the following natural question : Is it possible to remove k from the power, that is, is there a constant α such that for each k the isomorphism problem for k bounded bandwidth graphs can be solved in $O(n^{\alpha})$ time. If it is impossible to remove k from the power and furthermore the power, described by $f(k)$, is unbounded, then for the set of all graphs the isomorphism problem is not in **P**.

It is known that if a set of graphs L is of bounded bandwidth then L is chain-like graphs, namely there exists a constant k such that for any graph $H \in L$, H can be decomposed by a partitions of nodes $V_0, V_1, \cdots, V_m$ with following properties :

(1) $|V_i| \leq k$ for each $0 \leq i \leq m$ $V(H) = \bigcup_{0 \leq i \leq m} V_i$, $V_i \cap V_j = \emptyset$, $i \neq j$,

(2) there no exist edges $\{x, y\}$ for any $x \in V_i$, $y \in V_j$, $0 \leq i < j \leq m$, $j - i \geq 2$.

In this paper, we consider the isomorphism problem for chain-like graphs with the following additional condition :

(3) for the above V_i, the subgraph induced by V_i is connected.

In this paper, we show that the isomorphism problem for the chain-like graphs with the additional condition (3) can be solved in $O(n^3)$ time. If there exists a constant α such that the isomorphism problem without the condition (3) can be solved in $O(n^{\alpha})$, then the isomorphism problem for a set of bounded bandwidth graphs can be also solved in $O(n^{\alpha})$.

2 Preliminaries

We consider *finite undirected and connected graphs* without *loops* and without *multiple* edges. For a graph X, we denote the set of nodes in X by $V(X)$.

Definition 2.1 A set of graphs L is k *chain-like graphs* if for any graph $H \in L$, H can be decomposed by a partitions of nodes $V_0, V_1, \cdots, V_m$ such that

(1) $|V_i| \leq k$ for each $0 \leq i \leq m$, $V(H) = \bigcup_{0 \leq i \leq m} V_i$, $V_i \cap V_j = \emptyset$, $i \neq j$,

(2) there no exist edges $\{x, y\}$ for any $x \in V_i$, $y \in V_j$, $0 \leq i < j \leq m$, $j - i \geq 2$.
We call the list of the partitions $(V_0, V_1, \cdots, V_m)$ *partition list with bounded width k of H*.

A set of graphs L is *k chain-like graphs with connected condition* if L holds the above conditions (1), (2), and

(3) for each $0 \leq i \leq m$, the subgraph induced by V_i is connected.
A set of graphs L is *chain-like graphs* (*with connected condition*) if there exists a constant k such that L is k chain-like graphs (with connected condition, respectively).

Let H be a graph, u and v be nodes in $V(H)$, and $S = \{s_1, s_2, \cdots, s_n\}$ be a subset of $V(H)$. By $d_H(u, v)$ we denote the distance between u and v in H, and by $d_H(S, u)$ we denote $min\{d_H(s_1, u)$, $d_H(s_2, u)$, $\cdots$, $d_H(s_n, u)\}$. We denote the set of nodes $\{u \mid d = d_H(S, u)\}$ by $l_S^d(H)$. We call the list of the levels $(l_S^0(H), l_S^1(H), \cdots, l_S^m(H))$, denoted by $level_H(S)$, *the level list* (*with start set S*), where $m = \max_{u \in V(H)} d_H(S, u)$. We call m *the length of $level_H(S)$* and denote by $|level_H(S)|$ and we say $\max_{0 \leq i \leq m} |l_S^i(H)|$ *the width* (*of $level_H(S)$*).

Definition 2.2 For a graph H, a integer k is *the distancewidth of H* if $k = \min_{S \subseteq V(H)} \{j \mid j$ is the width of $level_H(S)\}$. Similarly, k is *the rooted distancewidth of H* if $k = \min_{u \in V(H)} \{j \mid j$ is the width of $level_H(\{u\})\}$.

Definition 2.3 A set of graphs L is *k bounded (rooted) distancewidth* if for any graph $H \in L$ the (rooted) distancewidth of H is at most k. A set of graphs L is *bounded (rooted) distancewidth* if there exists a constant k such that L is k bounded (rooted) distancewidth.

Definition 2.4 Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of sets of graphs. The class $\mathcal{C}_2$ *covers* the class $\mathcal{C}_1$, denoted by $\mathcal{C}_1 \prec \mathcal{C}_2$, if for any set $L_1 \in \mathcal{C}_1$, there exists a set $L_2 \in \mathcal{C}_2$ such that $L_1 \subseteq L_2$.

Example 2.1

Let $\mathcal{D}_e$ be the class $\{L \mid L$ is a set of graphs with bounded degree $\}$,
$\mathcal{T}$ be the class $\{L \mid L$ is a set of graphs with bounded treewidth $\}$,
$\mathcal{C}_u$ be the class $\{L \mid L$ is a set of graphs with bounded cutwidth $\}$,
$\mathcal{B}$ be the class $\{L \mid L$ is a set of graphs with bounded bandwidth $\}$,
$\mathcal{C}_h$ be the class $\{L \mid L$ is chain-like graphs $\}$, and
$\mathcal{D}_r$ be the class $\{L \mid L$ is bounded rooted distancewidth $\}$.

Then $\mathcal{D}_e \not\prec \mathcal{T}$ and $\mathcal{T} \not\prec \mathcal{D}_e$,
$\mathcal{C}_h \prec \mathcal{B}$ and $\mathcal{B} \prec \mathcal{C}_h$,
$\mathcal{D}_r \prec \mathcal{B} \prec \mathcal{C}_u \prec \mathcal{T}$, and
$\mathcal{D}_r \prec \mathcal{B} \prec \mathcal{C}_u \prec \mathcal{D}_e$.

(See [6] table 2 in p.550)

Proposition 2.1 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of sets of graphs, and assume that there exists a constant α such that for any $L \in \mathcal{C}_2$ the isomorphism problem for L can be solved in $O(n^\alpha)$. Then, $\mathcal{C}_2$ covers $\mathcal{C}_1$ implies that for any $L \in \mathcal{C}_1$ the isomorphism problem for L can be solved in $O(n^\alpha)$.*

3 Results

Rooted graphs X_{r_x} with root $r_x \in V(X)$ and Y_{r_y} with $r_y \in V(Y)$ are isomorphic if there exists a isomorphic bijection $f : V(Y) \to V(X)$ such that $f(y_r) = x_r$.

Lemma 3.1 *Let X and Y be k chain-like graphs and r_x and r_y be nodes in $V(X)$ and $V(Y)$ respectively. Then given the level list $level_X(r_x)$ with width k , and $level_Y(r_y)$ (it may be not level list with width k) as inputs, the decision whether the rooted graph X_{r_x} and Y_{r_y} are isomorphic can be solved in $O(|V(X)|)$.*

Proof. Let $m_x = |level_H(r_x)|$, $m_y = |level_H(r_y)|$, and $R_1, R_2, \cdots, R_{m_x}$ be sets of isomorphisms. By X_i and Y_i, we denote the induced subgraphs by $l^i_{r_x}(X)$ and $l^i_{r_y}(Y)$ respectively. The following procedure *sub-RCGI* work correctly in $O(|V(X)|)$

Procedure sub-RCGI($level_H(r_x), level_H(r_x)$)

  if $m_x \neq m_y$ **then** return false
  if $|X_i| \neq |Y_i|$ for some $1 \leq i \leq m_x$ **then** return false
  if X_i and Y_i are not isomorphic for some $1 \leq i \leq m_x$ **then** return false
  Comput all isomorphisms to X_i from Y_i for each $1 \leq i \leq m_x$
    (We say the isomorphisms $f^i_0, f^i_1, \cdots, f^i_{j_i}$)
  Initialize $R_1 := \{f^1_0, f^1_1, \cdots, f^1_{j_1}\}$ and $R_i := \emptyset$ for each $2 \leq i \leq m_x$
  for $i := 1$ **to** $m_x - 1$ **do**
    for each isomorphism $f^i_s \in R_i$
      if for all $u \in V(Y_i)$ and $v \in V(Y_{i+1})$,
        u and v are adjacent iff $f^i_s(u)$ and $f^{i+1}_t(v)$ are adjacent
        then add f^{i+1}_t in R_{i+1}.
  if $R_{m_x} \neq \emptyset$
    then return true
    else return false
end.

$\square$

Theorem 3.2 *Let L be a set of graphs with k bounded rooted distancewidth. If graphs X and Y are in L, then the decision whether X and Y are isomorphic can be solved in $O(|V(X)|^3)$ time.*

Proof. Let X and Y be graphs in L. From $X \in L$, X has a level list $level_X(x_i)$ with at most width k for some $x_i \in V(X)$.

Procedure RCGI

  Construct $level_X(x_i)$ for each $1 \leq i \leq n$ $(V(X) = \{x_1, \cdots, x_n\})$
  Find a $level_X(x_i)$ with at most width k and fix such x_i as x
  Construct $level_Y(y_i)$ for each $1 \leq i \leq n$ $(V(Y) = \{y_1, \cdots, y_n\})$
  for $i := 1$ **to** n **do**
    if sub-RCGI($level_X(x), level_Y(y_i)$) = true **then** return true
  return false
end.

For a graph H and a node u in $V(H)$, $level_H(u)$ can be constructed in $O(n^2)$ time. Thus total time is $O(n^3)$.

$\square$

Lemma 3.3 *Let $\mathcal{D}_r$ be the class $\{L \mid L$ is bounded rooted distancewidth $\}$. Let $\mathcal{C}_c$ be the class $\{L \mid L$ is chain-like graphs with connected condition$\}$. Then, $\mathcal{C}_c \prec \mathcal{D}_r$.*

Proof Let L be a set of graphs in $\mathcal{C}_c$ such that L is k chain-like graphs for some k, H be a graph in L, and $(V_0, V_1, \cdots, V_m)$ be a partition list of H with at most width k. Since $H \in L \in \mathcal{C}_c$, we can assume that the subgraph induced by V_i is connected. First we choice arbitrarily a node r in V_0, then we assign the distance from the root r to each node in H. We call the assigned distance the label for each node. Let s_i and l_i be the smallest and largest label of nodes in V_i respectively for each $1 \leq i \leq m$. To show this lemma, we need the following facts :

fact 1 : Since the subgraph induced by V_i is connected, $l_i - s_i \leq k - 1$.
fact 2 : From $s_{i+1} - s_i \geq 1$, $s_i + e \leq s_{i+e}$ for any integer e.

Let d be a label and let p (q) be the largest (smallest) integer such that for any $j < p$ $(q > j)$ V_j does not have a node with label d respectively. Now we will show that $q - p < k$, in other words, the number of the partitions which have a node with label d is at most k. Suppose, to the contrary, that $p + k \leq q$. Since there exists a node with label d in V_p and the fact 1, $d \leq l_p \leq s_p + k - 1$. From there exists a node with label d in V_q, the contrary assumption and the fact 2, $s_p + k \leq s_{p+k} \leq s_q \leq d$. From the contradiction that $d \leq s_p + k - 1 < s_p + k \leq d$, for any label d the number of the partitions which have a node with label d is at most k. Therefore for any label d there exist at most k^2 nodes which have the label d. This means that the rooted distancewidth of H is at most k^2. Let $L' \in \mathcal{D}_r$ be the set $\{H \mid H$ has at most k^2 rooted distancewidth $\}$. From $L \subseteq L'$, $\mathcal{C}_c \prec \mathcal{D}_r$. □

From Proposition 2.1, Theorem 3.2 and Lemma 3.3, we obtain the following main theorem.

Theorem 3.4 *Let k be a constant and L be a set of graph with the following properties : for any graph $H \in L$, H can be decomposed by a partitions of nodes $V_0, V_1, \cdots, V_m$ such that*

(1) $|V_i| \leq k$ for each $0 \leq i \leq m$, $V(H) = \bigcup_{0 \leq i \leq m} V_i$, $V_i \cap V_j = \emptyset$, $i \neq j$,

(2) there no exist edges $\{x, y\}$ for any $x \in V_i$, $y \in V_j$, $0 \leq i < j \leq m$, $j - i \geq 2$,

(3) the subgraph induced by V_i is connected for each $0 \leq i \leq m$.

If graphs X and Y are in L, then the decision whether X and Y are isomorphic can be solved in $O(|V(X)|^3)$.

Some pepole may hope that $\mathcal{C}_h \prec \mathcal{D}_r$, but it does not hold unfortunately.

Theorem 3.5 *Let $\mathcal{C}_h$ be the class $\{L \mid L$ is chain-like graphs$\}$ and $\mathcal{D}_r$ be the class $\{L \mid L$ is bounded rooted distancewidth$\}$. Then, $\mathcal{C}_h \not\prec \mathcal{D}_r$.*

Proof. To show this theorem, we will construct a set of graphs L with the following properties :

(1) L is 3 chain-like graphs,

(2) for each positive integer k, there exists a graph $H \in L$ such that the rooted distancewidth of H is more than k.

$L = \{H_2, H_3, H_4, \cdots\}$ is described in Fig.1. It is easy to see that for each $2 \leq k$ the rooted distancewidth of H_k is more than k.

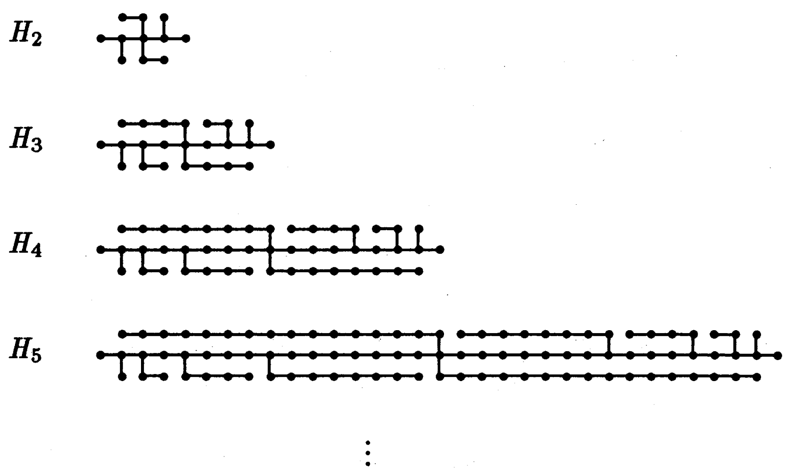

Fig.1 The graphs of L

□

4 Concluding Remarks

In this paper, we showed a isomorphism problem for a restriction chain-like graphs is solved in $O(n^3)$ time. Let $\mathcal{D}_i$ be the class $\{L \mid L$ is bounded distancewidth $\}$ and $\mathcal{D}_r$ be the class $\{L \mid L$ is bounded rooted distancewidth $\}$. Then, we conjecture that $\mathcal{D}_i \prec \mathcal{D}_r$.

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